RADIATION FROM A UNIFORMLY ROTATING CHARGE DISTRIBUTION IN A PLASMA IN A MAGNETIC FIELD

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RADIATION FROM A UNIFORMLY ROTATING CHARGE DISTRIBUTION IN A PLASMA IN A MAGNETIC FIELD

by

Stanley Gianzero*

INTRODUCTION

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The problem of energy radiated in a plasma from a uniformly rotating charge distribution is of importance in connection with the generation of high power microwave radiation. The dispersion properties of the plasma in a magnetic field may be used to confine the radiation either in the fundamental frequency of rotation of the charge or in a higher harmonic, in which case this radiation mechanism can be used as a frequency multiplier. A simple scheme for the plasma and the charge distribution is used in the present analysis to discuss the fundamental properties of the radiated spectrum. Author

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THEORETICAL ANALYSIS

This theoretical research has been devoted to the analysis of the spectrum of electromagnetic energy radiated by an electric charge which rotates at uniform velocity in a plasma. It is well known that a point charge rotating in a vacuum radiates a spectrum of lines which correspond to the harmonics of the angular frequency of motion. For nonrelativistic velocities of the charge the radiated energy is confined to the first few lines of the spectrum. For highly relativistic velocities the spectral distribution of the radiated energy increases first with the order of the harmonic, reaches a maximum and decreases thereafter.

A different behavior has to be expected if the charge rotates in a plasma in a constant magnetic field. In this case the medium is highly dispersive and a resonant situation has to be expected where the dominant part of the radiated energy should be confined to a particular harmonic of the angular frequency of the rotating charge.

In the present analysis a linear uniform distribution of electric charge has been considered parallel to the axis z of a cylindrical system of coordinates r, φ, z. The charge distribution rotates about the z axis with a constant angular velocity ω_o; and q_o, r_o denote the charge per unit length and the radius of the orbit respectively. In the frame of reference r, φ, z, the components of the electric current density $\vec{j}$ associated with the charge motion are

$$j_r = j_z = 0, \qquad j_{\varphi_o} = \frac{\omega_o q_o}{2\pi} \delta(r-r_o) \sum_{m=-\infty}^{+\infty} e^{-im\omega_o \tau} \tag{1}$$

where $\delta(r-r_o)$ is the Dirac δ-function and τ is related to the times t and the angular position φ:

$$\tau = t - \frac{\varphi}{\omega_o} \quad . \tag{2}$$

The charge distribution rotates in a uniform plasma in the presence of a uniform and constant magnetic field parallel to the x axis. An ideal plasma is considered where electron collisions are neglected. Assuming that the angular frequency ω_o is sufficiently large, the ion motion may be neglected. The current density $\vec{j}$ induces in the plasma an electron motion with components u_r, u_φ, and an electromagnetic field with finite components E_r, E_φ of the electric field and the component H_z of the magnetic field. Furthermore the intensity of the electromagnetic field is assumed to be small, such that the equations of motion of the electrons may be linearized. Thus the governing momentum and Maxwell equations in the plasma reduce to

$$\frac{\partial u_r}{\partial t} + \frac{q}{m_e} E_r + \omega_c u_\varphi = 0$$

$$\frac{\partial u_\varphi}{\partial t} + \frac{q}{m_e} E_\varphi - \omega_c u_r = 0$$

$$\frac{1}{r} \frac{\partial H_z}{\partial \varphi} - \varepsilon_o \frac{\partial E_r}{\partial t} + nqu_r = 0 \tag{3}$$

$$\frac{\partial H_z}{\partial r} + \varepsilon_o \frac{\partial E_\varphi}{\partial t} - nqu_\varphi = 0$$

$$\frac{1}{r} \frac{\partial}{\partial r} (rE_\varphi) - \frac{1}{r} \frac{\partial E_r}{\partial \varphi} + \mu_o \frac{\partial H_z}{\partial t} = 0$$

where q, m are the electron charge and mass respectively; n, ω_c are the electron density and the electron cyclotron frequency; ϵ_o, μ_o are the dielectric and magnetic permeabilities of a vacuum. With the exclusion of the electron pressure term in the momentum equations, the excitation of longitudinal modes in the plasma is not included in the present analysis.

A particular solution of system (3), which is consistent with the current density (1) is of the form:

$$\Psi(r,\varphi,t) = \sum_{m=-\infty}^{+\infty} \Psi_m(r)\, e^{-im\omega_o \tau} \tag{4}$$

where Ψ stands for each component of the electron velocity and electromagnetic field. In particular, E_{rm} and $E_{\varphi m}$ are given in terms of H_{zm} by the equations

$$\begin{aligned} E_{rm} &= -\frac{1}{\omega_o \epsilon_o \kappa_m^2}\left[\frac{1}{r} H_{zm} + \frac{1}{m^2}\frac{\omega_c}{\omega_o}\frac{\omega_p^2}{m^2\omega_o^2 - \omega_c^2 - \omega_p^2}\frac{dH_{zm}}{dr}\right] \\ E_{\varphi m} &= -\frac{i}{m\omega_o \epsilon_o \kappa_m^2}\left[\frac{\omega_c}{\omega_o}\frac{\omega_p^2}{m^2\omega_o^2 - \omega_c^2 - \omega_p^2}\frac{1}{r} H_{zm} + \frac{dH_{zm}}{dr}\right] \end{aligned} \tag{5}$$

where ω_p is the plasma frequency

$$\omega_p = \left(\frac{nq^2}{\epsilon_o m_e}\right) \tag{6}$$

and

$$\varkappa_m^2 = 1 - \frac{\omega_p^2}{m^2\omega_o^2} \frac{1}{1+\frac{\omega_c^2}{\omega_p^2 - m^2\omega_o^2}} \tag{7}$$

H_{zm} satisfies the equation

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dH_{zm}}{dr}\right) + \left(m^2\frac{\omega_o^2}{c^2}\varkappa_m^2 - \frac{m^2}{r^2}\right)H_{zm} = 0 \tag{8}$$

where $c = (\epsilon_o \mu_o)^{-\frac{1}{2}}$ is the speed of light in a vacuum. The solutions of Eq. (8) are

$$\begin{aligned} H_{zm} &= a_{im} J_m\left(\frac{m\omega_o \varkappa_m r}{c}\right), \quad (r < r_o) \\ H_{zm} &= a_{em} H_m^{(1)}\left(\frac{m\omega_o \varkappa_m r}{c}\right), \quad (r > r_o) \end{aligned} \tag{9}$$

where J_m, $H_m^{(1)}$ are the Benel and Hankel functions respectively. a_{im}, a_{em} are the integration constants to be determined with the boundary conditions at $r = r_o$. The boundary conditions at $r = r_o$ are

$$\begin{aligned} (E_\varphi^{\,i} - E_\varphi^{\,e})_{r=r_o} &= 0 \\ (H_z^{\,i} - H_z^{\,e})_{r=r_o} &= \lim_{\epsilon \to 0} \int_{r_o-\epsilon}^{r_o+\epsilon} j_\varphi \, dr \end{aligned} \tag{10}$$

where the superscripts i, e indicate the field components in the regions $r<r_o$ and $r>r_o$ respectively. The boundary conditions (10) yield

$$a_{im} J_m\left(\frac{m\omega_o \varkappa_m r_o}{c}\right) - a_{em} H_m^{(1)}\left(\frac{m\omega_o \varkappa_m r_o}{c}\right) = \frac{\omega_o q}{2\pi}$$

(11)

$$a_{im} J'_m\left(\frac{m\omega_o \varkappa_m r_o}{c}\right) - a_{em} H_m^{(1)\prime}\left(\frac{m\omega_o \varkappa_m r_o}{c}\right) = - \frac{\omega_c q_o c}{2\pi m\omega_o \varkappa_m r_o} \frac{\omega_p^2}{m^2\omega_o^2 - \omega_c^2 - \omega_p^2}$$

where J'_m and $H_m^{(1)\prime}$ are the derivatives of J_m, $H_m^{(1)}$ with respect to $m\omega_o \varkappa_m r/c$ for $r=r_o$. From (11) one obtains

$$a_{im} = - \frac{m\omega_o^2 q_o \varkappa_m r_o}{4c}\left[\frac{\omega_c c}{m\omega_o^2 \varkappa_m r_o} \frac{\omega_p^2}{m^2\omega_o^2 - \omega_c^2 - \omega_p^2} H_m^{(1)}\left(\frac{m\omega_o \varkappa_m r_o}{c}\right) + H_m^{(1)\prime}\left(\frac{m\omega_o \varkappa_m r_o}{c}\right)\right]$$

(12)

$$a_{em} = - \frac{m\omega_o^2 q_o \varkappa_m r_o}{4c}\left[\frac{\omega_c c}{m\omega_o^2 \varkappa_m r_o} \frac{\omega_p^2}{m^2\omega_o^2 - \omega_c^2 - \omega_p^2} J_m\left(\frac{m\omega_o \varkappa_m r_o}{c}\right) + J'_m\left(\frac{m\omega_o \varkappa_m r_o}{c}\right)\right]$$

To obtain the power radiated by the charge distribution per unit length, one can compute the flux of the Poynting vector across a cylinder of unit length and radius $r>r_o$ coaxial with the z axis. One obtains

$$W = \int_0^{2\pi} r E_\varphi^e H_z^e \, d\varphi =$$

(13)

$$= \frac{1}{2\varepsilon_o} \omega_o q_o^2 \sum_m \frac{m}{\varkappa_m^2} \left[\beta_m J'_m(m\beta_m) + \frac{\omega_c}{m\omega_o} \frac{\omega_p^2}{m^2\omega_o^2 - \omega_c^2 - \omega_p^2} J_m(m\beta_m)\right]^2$$

where the sum is extended to the positive values of m for which $\varkappa_m$ is a real quantity, and

$$\beta_m = \frac{\omega_o r_o}{c} \varkappa_m \qquad . \tag{14}$$

β_m is the ratio of the charge velocity $\omega_o r_o$ to the phase velocity $c/\varkappa_m$ of a plane electromagnetic wave of frequency $m\,\omega_o/2\pi$ which propagates perpendicular to the magnetic field.

The radiated energy (13) depends strongly upon the dispersion property of the plasma which is defined by the index of refraction $\varkappa_m$. For an arbitrary value $\omega/2\pi$ of the frequency of the electromagnetic field, the index of refraction $\varkappa$ is given by

$$\varkappa^2 = 1 - \frac{\omega_p^2}{\omega^2} \frac{1}{1 + \dfrac{\omega_c^2}{\omega_p^2 - \omega^2}} \tag{15}$$

In Fig. 1, $\varkappa^2$ is plotted versus the ratio $x = \omega/\omega_c$ for a given value of the parameter $\alpha = \omega_p/\omega_c$. The singularities are found at

$$x = 0, \qquad x = \sqrt{1+\alpha^2} \tag{16}$$

and $\varkappa$ vanishes at

$$x = \sqrt{\frac{1}{4} + \alpha^2} \pm \frac{1}{2} \tag{17}$$

Furthermore $\varkappa$ is larger than one in the range

$$\alpha < x < \sqrt{1+\alpha^2} \tag{18}$$

and $\varkappa$ is imaginary in the ranges

$$0 < x < \sqrt{\frac{1}{4}+\alpha^2} - \frac{1}{2} \quad , \quad \sqrt{1+\alpha^2} < x < \sqrt{\frac{1}{4}+\alpha^2} + \frac{1}{2} \tag{19}$$

Thus energy cannot be radiated in the ranges of frequency which satisfy conditions (19). On the other hand, a harmonic of the angular frequency ω_o may be found in the range (18) and it is possible to select a velocity $\omega_o r_o$ of the rotating charge distribution such that

$$\beta_m > 1 \tag{20}$$

Condition (20) means that the charge distribution rotates at a velocity $\omega_o r_o$ which is larger than the phase velocity of the m^{th} harmonic of the radiated field. When this condition is satisfied the Cerenkov radiation process contributes to the energy radiated in the m^{th} harmonic.

Assume for instance that the charge distribution rotates at the electron cyclotron frequency ($\omega_o = \omega_c$). One observes that the frequency range where $\varkappa > 1$ satisfies the condition

$$\sqrt{1+\alpha^2} - \alpha \leq 1 \tag{21}$$

Thus one harmonic alone can be found in the range (18). Two different situations can be found depending upon the ratio $\alpha = \omega_p/\omega_c$. Consider first the case $\alpha < 1$. Condition (20) can be satisfied for the first harmonic only ($m = 1$). The energy radiated in the first line of the spectrum is

$$\frac{\omega_o q_o^2}{2\epsilon_o \varkappa_1^2}\,[\beta_1 J_1'(\beta_1)-J_1(\beta_1)]^2 = \frac{1}{2}\,\mu_o \omega_c^3 r_o^2 q_o^2 J_2^2(\beta_1) \tag{22}$$

It is worthwhile pointing out that for $m = 1$, $\omega_o = \omega_c$ the index of refraction $\varkappa$ is

$$\varkappa_1 = \sqrt{2-\alpha^2} \tag{23}$$

Therefore the condition $\beta_1 > 1$ can be achieved only if the charge distribution rotates at relativistic velocities. A different situation is found when $\alpha > 1$. In particular, if α satisfies the condition

$$\alpha > \sqrt{2} \tag{24}$$

by virtue of (19) the lowest harmonics

$$m < \sqrt{\frac{1}{4}+\alpha^2} - \frac{1}{2} \tag{25}$$

cannot be radiated. Again, one harmonic may be found in the range (18) where the radiation can be particularly strong provided condition (20) is satisfied. Nevertheless one observes that for large values of α the range (18) becomes

$$\sqrt{1+\alpha^2} - \alpha \sim \frac{1}{2\alpha}$$

i.e., it decreases as α increases. Hence a situation $\beta_m > 1$ for $m >> 1$ becomes too critical to be of any practical interest.

In both cases $\alpha \gtrless 1$, for large values of m one has

$$\lim_{m \to \infty} \beta_m = \frac{\omega_o r_o}{c}$$

Thus for nonrelativistic velocities the energy radiated in the higher harmonics of the spectrum decreases rapidly as m increases.

TO EDITOR
Per Cond 20

CONCLUDING REMARKS

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The charge distribution, rotating at the cyclotron frequency, produces a strong radiation at the harmonic where condition (20) is satisfied. For relativistic velocities, condition (20) may be achieved at the fundamental harmonic ($m = 1$) provided the plasma frequency ω_p is smaller than the electron frequency ω_c. Conversely, when ω_p is large compared to ω_c the lower harmonics are suppressed and the main part of the radiated spectrum is confined to the harmonic of the order $m \sim \omega_p/\omega_c$.

~~Author~~

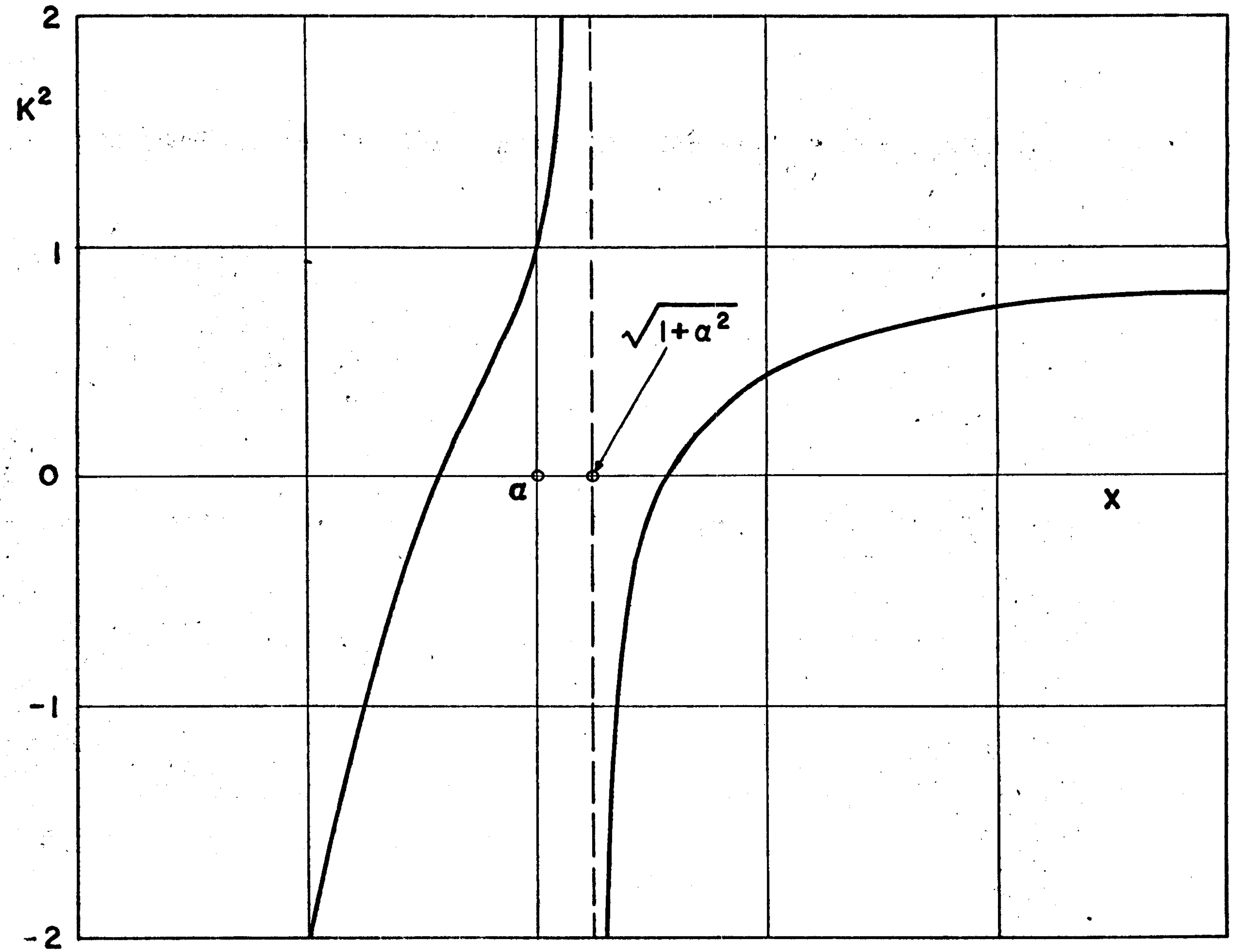

FIG. I INDEX OF REFRACTION OF THE PLASMA IN A MAGNETIC FIELD